QUADRATIC EQUATION

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CONTENTS

➤ BASIC CONCEPT

❖ Every algebraic polynomial of second degree is called a quadratic polynomial.

For Example :

(i) $3x^2 + 5x + 7$ (ii) $8x^2 - 6x$

(iii) $5x^2 - 7$ (iv) $\sqrt{2}\,x^2 + 6x - \sqrt{3}$

❖ The general form of quadratic polynomial is $ax^2 + bx + c$; where a, b, c are real numbers, $a \neq 0$ and x is variable.

For a particular quadratic polynomial the values of a, b, c are constant and for this reason a, b and c are also called real constants. For example, in quadratic polynomial $3x^2 - 5x + 8$; 3, – 5 and 8 are constant where as x is variable.

➤ VALUE OF A QUADRATIC POLYNOMIAL

The value of a quadratic polynomial $ax^2 + bx + c$

(i) at $x = \alpha$ is $a(\alpha)^2 + b(\alpha) + c = a\alpha^2 + b\alpha + c$

(ii) at $x = \beta$ is $a\beta^2 + b\beta + c$

(iii) at $x = 5$ is $a(5)^2 + b(5) + c = 25a^2 + 5a + c$

In the same way :

(i) Value of $5x^2 - 3x + 4$ at $x = 2$ is

$$= 5(2)^2 - 3(2) + 4$$
$$= 20 - 6 + 4 = 18$$

(ii) Value of $x^2 - 8x - 15$ at $x = -1$ is

$$= (-1)^2 - 8(-1) - 15$$
$$= 1 + 8 - 15 = -6$$

(iii) Value of $7x^2 - 4$ at $x = \frac{2}{3}$ is $= 7\left(\frac{2}{3}\right)^2 - 4$

$$= 7 \times \frac{4}{9} - 4$$
$$= \frac{28 - 36}{9} = \frac{8}{9}$$

ZEROS OF A QUADRATIC POLYNOMIAL

The value of the polynomial $x^2 - 7x + 10$ at :

(i) $x = 1$ is $(1)^2 - 7 \times 1 + 10 = 1 - 7 + 10 = 4$

(ii) $x = 2$ is $(2)^2 - 7 \times 2 + 10 = 4 - 14 + 10 = 0$

(iii) $x = 3$ is $(3)^2 - 7 \times 3 + 10 = 9 - 21 + 10 = -2$

(iv) $x = 5$ is $(5)^2 - 7 \times 5 + 10 = 25 - 35 + 10 = 0$

It is observed here that for $x = 2$ and $x = 5$; the value of polynomial $x^2 - 7x + 10$ is zero. These two values of x are called zeros of the polynomial.

Thus, if for $x = \alpha$, where α is a real number, the value of given quadratic polynomial is zero; the real number α is called zero of the quadratic polynomial.

❖ EXAMPLES ❖

Ex.1 Show that :

(i) $x = 3$ is a zero of quadratic polynomial $x^2 - 2x - 3$.

(ii) $x = -2$ is a zero of quadratic polynomial $3x^2 + 7x + 2$.

(iii) $x = 4$ is not a zero of quadratic polynomial $2x^2 - 7x - 5$.

Sol. (i) The value of $x^2 - 2x - 3$ at $x = 3$ is

$(3)^2 - 2 \times 3 - 3 = 9 - 6 - 3 = 0$

$\Rightarrow$ $x = 3$ is a zero of quadratic polynomial $x^2 - 2x - 3$.

(ii) The value of $3x^2 + 7x + 2$ at $x = -2$ is

$3(-2)^2 + 7(-2) + 2 = 12 - 14 + 2 = 0$

$\Rightarrow$ $x = -2$ is a zero of quadratic polynomial $3x^2 + 7x + 2$

(iii) The value of $2x^2 - 7x - 5$ at $x = 4$ is
$2(4)^2 - 7(4) - 5 = 32 - 28 - 5 = -1 \neq 0$

$\Rightarrow$ $x = 4$ is not a zero of quadratic polynomial $2x^2 - 7x - 5$.

Ex.2 Find the value of m, if $x = 2$ is a zero of quadratic polynomial $3x^2 - mx + 4$.

Sol. Since, $x = 2$ is a zero of $3x^2 - mx + 4$

$\Rightarrow$ $3(2)^2 - m \times 2 + 4 = 0$

$\Rightarrow$ $12 - 2m + 4 = 0$, i.e., $m = 8$.

QUADRATIC EQUATION & IT's ROOTS

Since, $ax^2 + bx + c$, $a \neq 0$ is a quadratic polynomial, $ax^2 + bx + c = 0$, $a \neq 0$ is called a quadratic equation.

(i) $-x^2 - 7x + 2 = 0$ is a quadratic equation, as $-x^2 - 7x + 2$ is a quadratic polynomial.

(ii) $5x^2 - 7x = 0$ is a quadratic equation.

(iii) $5x^2 + 2 = 0$ is a quadratic equation, but

(iv) $-7x + 2 = 0$ is not a quadratic equation.

❖ EXAMPLES ❖

Ex.3 Which of the following are quadratic equations, give reason :

(i) $x^2 - 8x + 6 = 0$

(ii) $3x^2 - 4 = 0$

(iii) $2x + \dfrac{5}{x} = x^2$

(iv) $x^2 + \dfrac{2}{x^2} = 3$

Sol. (i) Since, $x^2 - 8x + 6$ is a quadratic polynomial

$\Rightarrow$ $x^2 - 8x + 6 = 0$ is a quadratic equation.

(ii) $3x^2 - 4 = 0$ is a quadratic equation.

(iii) $2x + \dfrac{5}{x} = x^3$

$\Rightarrow$ $2x^2 + 5 = x^3$

$\Rightarrow$ $x^3 - 2x^2 - 5 = 0$; which is cubic and not a quadratic equation.

(iv) $x^2 + \dfrac{2}{x^2} = 3$

$\Rightarrow$ $x^4 + 2 = 2x^2$

$\Rightarrow$ $x^4 - 2x^2 + 2 = 0$; which is biquadratic and not a quadratic equation.

Ex.4 In each of the following, determine whether the given values are solutions (roots) of the equation or not :

(i) $3x^2 - 2x - 1 = 0$; $x = 1$

(ii) $x^2 + 6x + 5 = 0$; $x = -1$, $x = -5$

(iii) $x^2 + \sqrt{2}\,x - 4 = 0$; $x = \sqrt{2}$, $x = -2\sqrt{2}$

Sol. (i) $\because$ Value of $3x^2 - 2x - 1$ at $x = 1$ is

$3(1)^2 - 2(1) - 1 = 3 - 2 - 1 = 0 = \text{RHS}$

$\therefore$ $x = 1$ is a solution of the given equation.

(ii) For $x = -1$, L.H.S. $= (-1)^2 + 6(-1) + 5$

$= 1 - 6 + 5 = 0 =$ R.H.S.

$\Rightarrow$ $x = -1$ is a solution of the given equation

For $x = -5$, L.H.S. $= (-5)^2 + 6(-5) + 5$

$= 25 - 30 + 5 = 0 =$ R.H.S.

$\Rightarrow$ $x = -5$ is a solution of the given equation.

(iii) For $x = \sqrt{2}$, L.H.S. $= x^2 + \sqrt{2}\,x - 4$

$= (\sqrt{2})^2 + \sqrt{2}(\sqrt{2}) - 4$

$= 2 + 2 - 4 = 0$

$=$ R.H.S.

$\therefore$ $x = \sqrt{2}$ is a solution of the given equation

For $x = -2\sqrt{2}$,

L.H.S. $= (-2\sqrt{2})^2 + \sqrt{2} \times -2\sqrt{2} - 4$

$= 4 \times 2 - 2 \times 2 - 4 = 0$ R.H.S.

$\therefore$ $x = -2\sqrt{2}$ is a solution of the given equation.

➤ SOLVING A QUADRATIC EQUATION BY FACTORISATION

Since, $3x^2 - 5x + 2$ is a quadratic polynomial; $3x^2 - 5x + 2 = 0$ is a quadratic equation.

Also,

$3x^2 - 5x + 2 = 3x^2 - 3x - 2x + 2$ [Factorising]

$= 3x(x - 1) - 2(x - 1)$

$= (x - 1)(3x - 2)$

In the same way :

$3x^2 - 5x + 2 = 0 \Rightarrow 3x^2 - 3x - 2x + 2 = 0$

[Factorising L.H.S.]

$\Rightarrow (x - 1)(3x - 2) = 0$

i.e., $x - 1 = 0$ or $3x - 2 = 0$

$\Rightarrow$ $x = 1$ or $x = \frac{2}{3}$;

which is the solution of given quadratic equation.

In order to solve the given Quadratic Equation:

1. Clear the fractions and brackets, if given.
2. By transfering each term to the left hand side; express the given equation as ; $ax^2 + bx + c = 0$ or $a + bx + cx^2 = 0$
3. Factorise left hand side of the equation obtained (**the right hand side being zero**).
4. By putting each factor equal to zero; solve it.

❖ EXAMPLES ❖

Ex.5 Solve :

(i) $x^2 + 3x - 18 = 0$ (ii) $(x - 4)(5x + 2) = 0$

(iii) $2x^2 + ax - a^2 = 0$; where 'a' is a real number.

Sol. (i) $x^2 + 3x - 18 = 0$

$\Rightarrow x^2 + 6x - 3x - 18 = 0$

$\Rightarrow x(x + 6) - 3(x + 6) = 0$

i.e., $(x + 6)(x - 3) = 0 \Rightarrow x + 6 = 0$

or $x - 3 = 0$

$\Rightarrow x = -6$ or $x = 3$

$\therefore$ Roots of the given equation are : -6 and 3

(ii) $(x - 4)(5x + 2) = 0 \Rightarrow x - 4 = 0$

or $5x + 2 = 0$

$\Rightarrow x = 4$ or $x = -\frac{2}{5}$

(iii) $2x^2 + ax - a^2 = 0$

$\Rightarrow 2x^2 + 2ax - ax - a^2 = 0$

$\Rightarrow 2x(x + a) - a(x + a) = 0$

i.e., $(x + a)(2x - a) = 0$

$\Rightarrow x + a = 0$ or $2x - a = 0$

$\Rightarrow x = -a$ or $x = \frac{a}{2}$

Ex.6 Solve the following quadratic equations :

(i) $x^2 + 5x = 0$ (ii) $x^2 = 3x$

(iii) $x^2 = 4$

Sol. (i) $x^2 + 5x = 0 \Rightarrow x(x + 5) = 0$

$\Rightarrow$ $x = 0$ or $x + 5 = 0$

$\Rightarrow$ $x = 0$ or $x = -5$

(ii) $x^2 = 3x$

$\Rightarrow$ $x^2 - 3x = 0$

$\Rightarrow$ $x(x - 3) = 0$

$\Rightarrow$ $x = 0$ or $x = 3$

(iii) $x^2 = 4$

$\Rightarrow$ $x = \pm 2$

Ex.7 Solve the following quadratic equations :

(i) $7x^2 = 8 - 10x$

(ii) $3(x^2 - 4) = 5x$

(iii) $x(x + 1) + (x + 2)(x + 3) = 42$

Sol. (i) $7x^2 = 8 - 10x$

$\Rightarrow$ $7x^2 + 10x - 8 = 0$

$\Rightarrow$ $7x^2 + 14x - 4x - 8 = 0$

$\Rightarrow$ $7x(x + 2) - 4(x + 2) = 0$

$\Rightarrow$ $(x + 2)(7x - 4) = 0$

$\Rightarrow$ $x + 2 = 0$ or $7x - 4 = 0$

$\Rightarrow$ $x = -2$ or $x = \frac{4}{7}$

(ii) $3(x^2 - 4) = 5x$

$\Rightarrow$ $3x^2 - 5x - 12 = 0$

$\Rightarrow$ $3x^2 - 9x + 4x - 12 = 0$

$\Rightarrow$ $3x(x - 3) + 4(x - 3) = 0$

$\Rightarrow$ $(x - 3)(3x + 4) = 0$

$\Rightarrow$ $x - 3 = 0$ or $3x + 4 = 0$

$\Rightarrow$ $x = 3$ or $x = -\frac{4}{3}$

(iii) $x(x + 1) + (x + 2)(x + 3) = 42$

$\Rightarrow$ $x^2 + x + x^2 + 3x + 2x + 6 - 42 = 0$

$\Rightarrow$ $2x^2 + 6x - 36 = 0$

$\Rightarrow$ $x^2 + 3x - 18 = 0$

$\Rightarrow$ $x^2 + 6x - 3x - 18 = 0$

$\Rightarrow$ $x(x + 6) - 3(x + 6) = 0$

$\Rightarrow$ $(x + 6)(x - 3) = 0$

$\Rightarrow$ $x = -6$ or $x = 3$

Ex.8 Solve for x : $12\,abx^2 - (9a^2 - 8b^2)\,x - 6ab = 0$

Sol. Given equation is :

$12abx^2 - 9a^2x + 8b^2x - 6ab = 0$

$\Rightarrow$ $3ax(4bx - 3a) + 2b(4bx - 3a) = 0$

$\Rightarrow$ $(4bx - 3a)(3ax + 2b) = 0$

$\Rightarrow$ $4bx - 3a = 0$ or $3ax + 2b = 0$

$\Rightarrow$ $x = \frac{3a}{4b}$ or $x = -\frac{2b}{3a}$

SOLVING A QUADRATIC EQUATION BY COMPLETING THE SQUARE

Every quadratic equation can be converted in the form :

$(x + a)^2 - b^2 = 0$ or $(x - a)^2 - b^2 = 0$.

Steps :

1. Bring, if required, all the term of the quadratic equation to the left hand side.
2. Express the terms containing x as $x^2 + 2xy$ or $x^2 - 2xy$.
3. Add and subtract y^2 to get $x^2 + 2xy + y^2 - y^2$ or $x^2 - 2xy + y^2 - y^2$; which gives $(x + y)^2 - y^2$ or $(x - y)^2 - y^2$.

Thus,

(i) $x^2 + 8x = 0$ $\Rightarrow$ $x^2 + 2x \times 4 = 0$

$\Rightarrow$ $x^2 + 2x \times 4 + 4^2 - 4^2 = 0$

$\Rightarrow$ $(x + 4)^2 - 16 = 0$

(ii) $x^2 - 8x = 0$ $\Rightarrow$ $x^2 - 2 \times x \times 4 = 0$

$\Rightarrow$ $x^2 - 2 \times x \times 4 + 4^2 - 4^2 = 0$

$\Rightarrow$ $(x - 4)^2 - 16 = 0$

❖ EXAMPLES ❖

Ex.9 Find the roots of the following quadratic equations (if they exist) by the method of completing the square.

(i) $2x^2 - 7x + 3 = 0$ (ii) $4x^2 + 4\sqrt{3}\,x + 3 = 0$

(iii) $2x^2 + x + 4 = 0$

Sol. (i) $2x^2 - 7x + 3 = 0$ $\Rightarrow x^2 - \frac{7}{2}x + \frac{3}{2} = 0$

[Dividing each term by 2]

$$\Rightarrow x^2 - 2 \times x \times \frac{7}{4} + \frac{3}{2} = 0$$

$$\Rightarrow x^2 - 2 \times x \times \frac{7}{4} + \left(\frac{7}{4}\right)^2 - \left(\frac{7}{4}\right)^2 + \frac{3}{2} = 0$$

$$\Rightarrow \left(x - \frac{7}{4}\right)^2 - \frac{49}{16} + \frac{3}{2} = 0$$

$$\Rightarrow \left(x - \frac{7}{4}\right)^2 - \left(\frac{49-24}{16}\right) = 0$$

$$\Rightarrow \left(x - \frac{7}{4}\right)^2 - \frac{25}{16} = 0$$

i.e., $\left(x - \frac{7}{4}\right)^2 = \frac{25}{16}$ $\quad \Rightarrow x - \frac{7}{4} = \pm \frac{5}{4}$

i.e., $x - \frac{7}{4} = \frac{5}{4} =$ $\quad$ or $\quad x - \frac{7}{4} = -\frac{5}{4}$

$\Rightarrow x = \frac{7}{4} + \frac{5}{4}$ $\quad$ or $x = \frac{7}{4} - \frac{5}{4}$

$\Rightarrow x = 3$ $\quad$ or $x = \frac{1}{2}$

(ii) $4x^2 + 4\sqrt{3}\,x + 3 = 0$

$$\Rightarrow x^2 + \sqrt{3}\,x + \frac{3}{4} = 0$$

i.e., $x^2 + 2 \times x \times \frac{\sqrt{3}}{2} + \left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}\right)^2 + \frac{3}{4} = 0$

$$\Rightarrow \left(x + \frac{\sqrt{3}}{2}\right)^2 - \frac{3}{4} + \frac{3}{4} = 0$$

i.e., $\left(x + \frac{\sqrt{3}}{2}\right)^2 = 0$

$\Rightarrow x + \frac{\sqrt{3}}{2} = 0$ $\quad$ and $\quad x = \frac{-\sqrt{3}}{2}$

$\therefore$ Roots are : $\frac{-\sqrt{3}}{2}$ and $\frac{-\sqrt{3}}{2}$

(iii) $2x^2 + x + 4 = 0$ $\quad \Rightarrow x^2 + \frac{x}{2} + 2 = 0$

i.e., $x^2 + 2 \times x \times \frac{1}{4} + \left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^2 + 2 = 0$

$$\Rightarrow \left(x + \frac{1}{4}\right)^2 - \frac{1}{16} + 2 = 0$$

$$\Rightarrow \left(x + \frac{1}{4}\right)^2 + \frac{31}{16} = 0$$

$$\left[-\frac{1}{16} + 2 = \frac{-1+32}{16} = \frac{31}{16}\right]$$

i.e., $\left(x + \frac{1}{4}\right)^2 = -\frac{31}{16}$

This is not possible as the square of a real number can not be negative.

SOLVING A QUADRATIC EQUATION BY USING QUADRATIC FORMULA

Hindu Method (Sri Dharacharya Method) :

By completing the perfect square as

$$ax^2 + bx + c = 0 \Rightarrow x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Adding and substracting $\left(\frac{b}{2a}\right)^2$

$$\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right] = 0$$

Which gives, $\quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Hence the Quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0)$ has two roots, given by

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Note : Every quadratic equation has at most two and only two real roots.

❖ EXAMPLES ❖

Ex.10 Solve the following quadratic equations by using quadratic formula :

(i) $x^2 - 7x + 12 = 0$

(ii) $3x^2 - x - 10 = 0$

Sol. (i) Comparing the given equation $x^2 - 7x + 12 = 0$ with standard quadratic equation $ax^2 + bx + c = 0$; we get : a = 1, b = – 7 and c = 12

$\therefore \quad b^2 - 4ac = (-7)^2 - 4 \times 1 \times 12$

$= 49 - 48 = 1$

and $\sqrt{b^2 - 4ac} = \sqrt{1} = 1$

$\because \quad x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\Rightarrow \quad x = \dfrac{7 \pm 1}{2 \times 1} = \dfrac{7+1}{2}$ or $\dfrac{7-1}{2} = 4$ or 3

(ii) Comparing the given equation

$3x^2 - x - 10 = 0$ with equation

$ax^2 + bx + c = 0$; we get : $a = 3$, $b = -1$ and $c = -10$

$\therefore \quad b^2 - 4ac = (-1)^2 - 4 \times 3 \times -10 = 1 + 120 = 121$

and $\sqrt{b^2 - 4ac} = \sqrt{121} = 11$

Hence, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\Rightarrow \quad x = \dfrac{1 \pm 11}{2} = \dfrac{1+11}{2}$ or $\dfrac{1-11}{2} = 6$ or -5

Ex.11 For a quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, prove that : $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Sol. $ax^2 + bx + c = 0$

$\Rightarrow 4a^2x^2 + 4abx + 4ac = 0$

[Multiplying by '4a']

$\Rightarrow (2ax)^2 + 2 \times 2ax \times b + b^2 - b^2 + 4ac = 0$

$\Rightarrow (2ax + b)^2 - b^2 + 4ac = 0$

$\Rightarrow (2ax + b)^2 = b^2 - 4ac$

$\Rightarrow 2ax + b = \pm \sqrt{b^2 - 4ac}$

$\Rightarrow 2ax = -b \pm \sqrt{b^2 - 4ac}$

$\Rightarrow x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Ex.12 Solve, by using quadratic formula, each of the following equations :

(i) $2x^2 + 5\sqrt{3}\,x + 6 = 0$

(ii) $3x^2 + 2\sqrt{5}\,x - 5 = 0$

Sol. (i) Comparing $2x^2 + 5\sqrt{3}\,x + 6 = 0$ with

$ax^2 + bx + c = 0$, we get :

$a = 2$, $b = 5\sqrt{3}$ and $c = 6$

$b^2 - 4ac = (5\sqrt{3})^2 - 4 \times 2 \times 6$

$= 25 \times 3 - 48 = 27$

$\sqrt{b^2 - 4ac} = \sqrt{27} = \sqrt{3 \times 3 \times 3} = 3\sqrt{3}$

$\therefore \quad x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{-5\sqrt{3} \pm 3\sqrt{3}}{2 \times 4}$

$= \dfrac{-5\sqrt{3} + 3\sqrt{3}}{4}$ or $\dfrac{-5\sqrt{3} - 3\sqrt{3}}{4}$

$= \dfrac{-2\sqrt{3}}{4}$ or $\dfrac{-8\sqrt{3}}{4} = -\dfrac{\sqrt{3}}{2}$ or $-2\sqrt{3}$

(ii) Comparing $3x^2 + 2\sqrt{5}\,x - 5 = 0$ with $ax^2 + bx + c = 0$, we get :

$a = 3$, $b = 2\sqrt{5}$ and $c = -5$

$b^2 - 4ac = (2\sqrt{5})^2 - 4 \times 3 \times -5 = 4 \times 5 + 60 = 80$

$\sqrt{b^2 - 4ac} = \sqrt{80} = \sqrt{16 \times 5} = 4\sqrt{5}$

$\therefore \quad x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{-2\sqrt{5} \pm 4\sqrt{5}}{2 \times 3}$

$= \dfrac{-2\sqrt{5} + 4\sqrt{5}}{6}$ or $\dfrac{-2\sqrt{5} - 4\sqrt{5}}{6}$

$= \dfrac{2\sqrt{5}}{6}$ or $\dfrac{-6\sqrt{5}}{6} = \dfrac{\sqrt{5}}{3}$ or $-\sqrt{5}$

Ex.13 Using the quadratic formula, solve the equation: $a^2b^2x^2 - (4b^2 - 3a^4)x - 12a^2b^2 = 0$

Sol. Comparing given equation with

$Ax^2 + Bx + C = 0$, we get :

$A = a^2b^2$, $B = -(4b^4 - 3a^4)$ and $C = -12a^2b^2$

$\therefore B^2 - 4AC = (4b^4 - 3a^4)^2 - 4 \times a^2b^2 \times (-12a^2b^2)$

$= 16b^8 + 9a^8 - 24a^4b^4 + 48a^4b^4$

$= 16b^8 + 9a^8 + 24a^4b^4 = (4b^4 + 3a^4)^2$

$\sqrt{B^2 - 4AC} = 4b^4 + 3a^4$

$\therefore x = \dfrac{-B \pm \sqrt{B^2 - 4AC}}{2A}$

$$= \frac{(4b^4 - 3a^4) \pm (4b^2 + 3a^4)}{2 \times a^2 b^2}$$

$$= \frac{4b^4 - 3a^4 + 4b^4 + 3a^4}{2a^2b^2}$$

$$\text{or } \frac{4b^4 - 3a^4 - 4b^4 - 3a^4}{2a^2b^2}$$

$$= \frac{8b^4}{2a^2b^2} \text{ or } \frac{-6a^4}{2a^2b^2} = \frac{4b^2}{a^2} \text{ or } \frac{-3a^2}{b^2}$$

➢ NATURE OR CHARACTER OF THE ROOTS OF A QUADRATIC EQUATION

The nature of the roots depends on the value of $b^2 - 4ac$. $b^2 - 4ac$ is called the **discriminant** of the quadratic equation $ax^2 + bx + c = 0$ and is generally, denoted by D.

$\therefore$ $D = b^2 - 4ac$

❖ **If D > 0**, i..e., $b^2 - 4ac > 0$, i.e., $b^2 - 4ac$ is positive; **the roots are real and unequal**. Also,

(i) If $b^2 - 4ac$ is a perfect square, the roots are rational and unequal.

(ii) If $b^2 - 4ac$ is positive but not perfect square, the roots are irrational and unequal.

❖ If D = 0, i.e., $b^2 - 4ac = 0$; **the roots are real and equal**.

❖ **If D < 0**, i.e., $b^2 - 4ac < 0$; i.e., $b^2 - 4ac$ is negative; the roots are not real, i.e., **the roots are imaginary**.

❖ EXAMPLES ❖

Ex.14 Without solving, examine the nature of roots of the equations :

(i) $2x^2 + 2x + 3 = 0$

(ii) $2x^2 - 7x + 3 = 0$

(iii) $x^2 - 5x - 2 = 0$

(iv) $4x^2 - 4x + 1 = 0$

Sol. (i) Comparing $2x^2 + 2x + 3 = 0$

with $ax^2 + bx + c = 0$; we get : a = 2, b = 2 and c = 3

$D = b^2 - 4ac = (2)^2 - 4 \times 2 \times 3 = 4 - 24$

$= -20$; which is negative.

$\therefore$ The roots of the given equation are imaginary.

(ii) Comparing $2x^2 - 7x + 3 = 0$

with $ax^2 + bx + c = 0$;

we get : a = 2, b = – 7 and c = 3

$D = b^2 - 4ac = (-7)^2 - 4 \times 2 \times 3$

$= 49 - 24 = 25$, which is perfect square.

$\therefore$ The roots of the given equation are rational and unequal.

(iii) Comparing $x^2 - 5x - 2 = 0$

with $ax^2 + bx + c = 0$;

we get : a = 1, b = – 5 and c = – 2

$D = b^2 - 4ac = (-5)^2 - 4 \times 1 \times -2$

$= 25 + 8 = 33$; which is positive but not a perfect square.

$\therefore$ The roots of the given equation are irrational and unequal.

(iv) Comparing $4x^2 - 4x + 1 = 0$

with $ax^2 + bx + c = 0$;

we get : a = 4, b = – 4, and c = 1

$D = b^2 - 4ac = (-4)^2 - 4 \times 4 \times 1$

$= 16 - 16 = 0$

$\therefore$ Roots are real and equal.

Ex.15 For what value of m, are the roots of the equation $(3m + 1)x^2 + (11 + m)x + 9 = 0$ equal?

Sol. Comparing the given equation

with $ax^2 + bx + c = 0$;

we get : a = 3m + 1, b = 11 + m and c = 9

$\therefore$ Discriminant, $D = b^2 - 4ac$

$= (11 + m)^2 - 4(3m + 1) \times 9$

$= 121 + 22m + m^2 - 108m - 36$

$= m^2 - 86m + 85$

$= m^2 - 85m - m + 85$

$= m(m - 85) - 1(m - 85)$

$= (m - 85)(m - 1)$

Since the roots are equal, D = 0

$\Rightarrow$ $(m - 85)(m - 1) = 0$

$\Rightarrow$ $m - 85 = 0$ or $m - 1 = 0$

$\Rightarrow$ $m = 85$ or $m = 1$

➤ SUM AND PRODUCT OF THE ROOTS

Let α and β be the two roots of the quadratic equation $ax^2 + bx + c = 0$.

Since, $ax^2 + bx + c = 0 \Rightarrow x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

then, let : $\alpha = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$

and $\beta = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$

$\therefore$ The sum of the roots $= \alpha + \beta$

$$= \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-b + \sqrt{b^2 - 4ac} - b - \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2b}{2a} = -\frac{b}{a}$$

And, the product of the roots $= \alpha \,.\, \beta$

$$= \left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)$$

$$= \frac{(-b)^2 - (\sqrt{b^2 - 4ac})^2}{4a^2} = \frac{b^2 - (b^2 - 4ac)}{4a^2}$$

$$= \frac{b^2 - b^2 + 4ac}{4a^2}$$

$$= \frac{4ac}{4a^2} = \frac{c}{a}$$

$\therefore$ If α and β are the roots of the quadratic equation $ax^2 + bx + c = 0$; then :

(i) The sum of the roots

$$= \alpha + \beta = -\frac{b}{a}$$

$$= -\frac{\text{coefficient of } x}{\text{coefficient of } x^2}$$

(ii) The product of the roots

$$= \alpha\beta = \frac{c}{a} = \frac{\text{constant term}}{\text{coefficient of } x^2}$$

➤ TO CONSTRUCT A QUADRATIC EQUATION WHOSE ROOTS ARE GIVEN

x^2 – (sum of roots) x + product of roots = 0

To get the quadratic equation with given roots :

(i) Find the sum of the roots.

(ii) Find the product of the roots.

(iii) Substitute the values of steps (i) and (ii) in

x^2 – (sum of the roots)x + (product of roots) = 0

and get the required quadratic equation.

❖ EXAMPLES ❖

Ex.16 For each quadratic equation given below, find the sum of the roots and the product of the roots :

(i) $x^2 + 3x - 6 = 0$ (ii) $2x^2 + 5\sqrt{3}\,x + 6 = 0$

(iii) $3x^2 + 2\sqrt{5}\,x - 5 = 0$

Sol. (i) Comparing $x^2 + 3x - 6 = 0$

with $ax^2 + bx + c = 0$,

we get :

$a = 1$, $b = 3$ and $c = -6$

$\therefore$ The sum of the roots $= -\dfrac{b}{a} = -\dfrac{3}{2}$

And, the product of the roots $= \dfrac{c}{a} = \dfrac{-6}{1} = -6$

(ii) Comparing $2x^2 + 5\sqrt{3}\,x + 6 = 0$ with $ax^2 + bx + c = 0$; we get :

$a = 2$, $b = 5\sqrt{3}$ and $c = 6$

$\therefore$ The sum of the roots $= -\dfrac{b}{a} = -\dfrac{5\sqrt{3}}{2}$

And, the product of the roots $= \dfrac{c}{a} = \dfrac{6}{2} = 3$

(iii) Comparing $3x^2 + 2\sqrt{5}\,x - 5 = 0$

with $ax^2 + bx + c = 0$; we get :

$a = 3$, $b = 2\sqrt{5}$ and $c = -5$

$\therefore$ The sum of the roots $= -\dfrac{b}{a} = -\dfrac{2\sqrt{5}}{3}$

and, the product of the roots $= \dfrac{c}{a} = \dfrac{-5}{3}$

Ex.17 Construct the quadratic equation whose roots are given below -

(i) 3, – 3

(ii) $3+\sqrt{3}$, $3-\sqrt{3}$

(iii) $\frac{2+\sqrt{5}}{2}$, $\frac{2-\sqrt{5}}{2}$

Sol. (i) Since, the sum of the roots

$= (3) + (-3) = 3 - 3 = 0$

and, the product of the roots

$= (3)\,(-3) = -9$

$\therefore$ The required quadratic equation is :

x^2 – (sum of roots) x + (product of roots) = 0

$\Rightarrow\ x^2 - (0)\,x + (-9) = 0$, i.e., $x^2 - 9 = 0$

(ii) Since, the sum of the roots

$= 3 + \sqrt{3} + 3 - \sqrt{3} = 6$

and, the product of the roots

$= (3+\sqrt{3})\,(3-\sqrt{3}) = 9 - 3 = 6$

$\therefore$ The required quadratic equation is :

x^2 – (sum of roots) x + (product of roots) = 0

$\Rightarrow\ x^2 - 6x + 6 = 0$

(iii) Since, the sum of the roots

$$= \frac{2+\sqrt{5}}{2} + \frac{2-\sqrt{5}}{2} = \frac{2+\sqrt{5}+2-\sqrt{5}}{2} = \frac{4}{2} = 2$$

and, the product of the roots

$$= \left(\frac{2+\sqrt{5}}{2}\right)\left(\frac{2-\sqrt{5}}{2}\right) = \frac{4-5}{4} - \frac{1}{4}$$

$\therefore$ The required quadratic equation is :

x^2 – (sum of roots) x + (product of roots) = 0

$$\Rightarrow\ x^2 - 2x + \left(-\frac{1}{4}\right) = 0$$

$$\Rightarrow\ x^2 - 2x - \frac{1}{4} = 0,$$

i.e., $4x^2 - 8x - 1 = 0$

Ex.18 If a and c are such that the quadratic equation $ax^2 - 5x + 3 = 0$ has 10 as the sum of the roots and also as the product of the roots, find a and c.

Sol. For $ax^2 - 5x + c = 0$

the sum of roots $= -\frac{\text{coefficient of } x}{\text{coefficient of } x^2}$

$$= -\frac{-5}{a} = \frac{5}{a}$$

and the product of roots

$$= \frac{\text{constant term}}{\text{coefficient of } x^2} = \frac{c}{a}$$

Given : The sum of the roots = 10

$\Rightarrow\ \frac{5}{a} = 10$, i.e., $10a = 5 \Rightarrow a = \frac{5}{10} = \frac{1}{2}$

The product of roots = 10

$\Rightarrow\ \frac{c}{a} = 10 \Rightarrow c = 10a = 10 \times \frac{1}{2} = 5$

$\Rightarrow\ a = \frac{1}{2}$ and $c = 5$

Ex.19 If one of the roots of the quadratic equation $2x^2 + px + 4 = 0$ is 2, find the the value of p. also find the value of the other roots.

Sol. As, 2 is one of the roots, x = 2 will satisfy the equation $2x^2 + px + 4 = 0$

$\Rightarrow\ 2(2)^2 + p(2) + 4 = 0$

$\Rightarrow\ 8 + 2p + 4 = 0$

i.e., $2p = -12$ and $p = -6$

Substituting $p = -6$ in the equation

$2x^2 + px + 4 = 0$; we get : $2x^2 - 6x + 4 = 0$

$\Rightarrow\ x^2 - 3x + 2 = 0$

[Dividing each term by 2]

$\Rightarrow\ x^2 - 2x - x + 2 = 0$

$\Rightarrow\ x(x-2)\,(x-1) = 0$

$\Rightarrow\ x - 2 = 0$ or $x - 1 = 0$

$\Rightarrow\ x = 2$ or $x = 1$

$\therefore$ The other (second) root is 1.

Ex.20 In the following, find the value (s) of p so that the given equation has equal roots.

(i) $3x^2 - 5x + p = 0$

(ii) $2px^2 - 8x + p = 0$

Sol. (i) Comparing $3x^2 - 5x + p = 0$

with $ax^2 + bx + c = 0$,

we get : $a = 3, b = -5$ and $c = p$

Since, the roots are equal ; the discriminant

$b^2 - 4ac = 0$

i.e., $(-5)^2 - 4 \times 3 \times p = 0$

$\Rightarrow$ $25 - 12p = 0$ and $p = \frac{25}{12} = 2\frac{1}{12}$

(ii) Comparing $2px^2 - 8x + p = 0$

with $ax^2 + bx + c = 0$;

we get : $a = 2p, b = -8$ and $c = p$

$b^2 - 4ac = 0$ [Given, that the roots are equal]

$\Rightarrow$ $(-8)^2 - 4 \times 2p \times p = 0$

$\Rightarrow$ $64 - 8p^2 = 0$

$\Rightarrow$ $-8p^2 = -64$, $p^2 = 8$ and $p = \pm\sqrt{8}$

i.e., $p = \pm 2\sqrt{2}$

➢ EQUATIONS REDUCIBLE TO QUADRATIC EQUATIONS

Type 1 : Equations of the form $ax^4 + bx^2 + c = 0$,

Method : Substitute $x^2 = y$ and solve.

❖ EXAMPLES ❖

Ex.22 Solve the following equations :

(i) $x^4 - 26x^2 + 25 = 0$ (ii) $z^4 - 10z^2 + 9 = 0$

Sol. (i) Substituting $x^2 = y$:

$x^4 - 26x^2 + 25 = 0$

$\Rightarrow$ $y^2 - 26y + 25 = 0$

i.e., $y^2 - 25y - y + 25 = 0$

$\Rightarrow$ $y(y - 25) - 1(y - 25) = 0$

i.e., $(y - 25)(y - 1) = 0$

$\Rightarrow$ $y - 25 = 0$ or $y - 1 = 0$

i.e., $y = 25$ or $y = 1$

$y = 25 \Rightarrow x^2 = 25$ | $y = 1 \Rightarrow x^2 = 1$

$\Rightarrow x = \pm 5$ | $\Rightarrow x = \pm 1$

$\therefore$ Roots of the given equation are : $\pm 5, \pm 1$

(ii) Substituting $z^2 = x$

$z^4 - 10z^2 + 9 = 0 \Rightarrow x^2 - 10x + 9 = 0$

i.e., $x^2 - 9x - x + 9 = 0$

$\Rightarrow$ $x(x - 9) - 1(x - 9) = 0$

i.e., $(x - 9)(x - 1) = 0$

$\Rightarrow$ $x - 9 = 0$ or $x - 1 = 0$

$x = 9 \Rightarrow z^2 = 9$ | $x = 1 \Rightarrow z^2 = 1$

$\Rightarrow z = \pm 3$ | $\Rightarrow z = \pm 1$

$\therefore$ Solution of the given equation is : $\pm 3, \pm 1$.

Type 2 : Equation of the form : $px + \frac{q}{x} = r$

Method : (i) Multiply each term by x.

(ii) Solve the quadratic equation obtained to get the non-zero value(s) of x.

Ex.23 Solve :

(i) $x + \frac{5}{x} = 6$ (ii) $3y + \frac{5}{16y} = 2$

Sol. (i) $x + \frac{5}{x} = 6$

$\Rightarrow$ $x^2 + 5 = 6x$ [Multiplying each term by x]

$\Rightarrow$ $x^2 - 6x + 5 = 0 \Rightarrow x^2 - 5x - x + 5 = 0$

i.e., $x(x - 5) - 1(x - 5) = 0$

$\Rightarrow$ $(x - 5)(x - 1) = 0$ i.e., $x - 5 = 0$

or $x - 1 = 0 \Rightarrow x = 5$ or $x = 1$.

$\therefore$ Required solution is 5, 1

(ii) $3y + \frac{5}{16y} = 2$

$\Rightarrow$ $3y \times 16y + 5 = 2 \times 16y$

$\Rightarrow$ $48y^2 - 32y + 5 = 0$

$\Rightarrow$ $48y^2 - 12y - 20y + 5 = 0$

i.e., $12y(4y - 1) - 5(4y - 1) = 0$

$\Rightarrow$ $(4y - 1)(12y - 5) = 0$

i.e., $4y - 1 = 0$ or $12y - 5 = 0$

$\Rightarrow$ $4y = 1$ or $12y = 5$ i.e., $y = \frac{1}{4}$ or $y = \frac{5}{12}$

$\therefore$ Required solutions is : $\frac{1}{4}, \frac{5}{12}$

Type 3 :

Equations involving one radical :

$$\sqrt{a - x^2} = bx + c$$

Method :

1. Square both the sides to get :
$$a - x^2 = (bx + c)^2$$
2. Now simplify it to get a quadratic equation.
3. Solve the quadratic equation obtained.

Ex.24 Solve :

(i) $\sqrt{x} + 2x = 1$ (ii) $\sqrt{3x^2 - 2} + 1 = 2x$

(iii) $\sqrt{2x^2 + 9} + x = 13$

Sol. (i) $\sqrt{x} + 2x = 1 \Rightarrow \sqrt{x} = 1 - 2x$

i.e., $x = (1 - 2x)^2$

$\Rightarrow x = 1 + 4x^2 - 4x$

i.e., $1 + 4x^2 - 4x - x = 0$

$\Rightarrow 4x^2 - 5x + 1 = 0$ i.e., $4x^2 - 4x - x + 1 = 0$

$\Rightarrow 4x(x - 1) - 1(x - 1) = 0$

i.e., $(x - 1)(4x - 1) = 0$

$\Rightarrow x - 1 = 0$

or $4x - 1 = 0$

i.e., $x = 1$ or $x = \frac{1}{4}$

➤ PROBLEMS ON QUADRATIC EQUATIONS

For solving problems based on quadratic equations, the following steps must be adopted :

1. Read the given statement of the problem carefully to find the required unknown quantity.
2. Take the unknown quantity as 'x' and according to the given statement, form an equation in terms of 'x'.
3. Simplify and solve the equation to get the value/values of 'x'.

❖ EXAMPLES ❖

Ex.25 Find two consecutive natural numbers, whose product is equal to 20.

Sol. Let the required two consecutive natural numbers be x and x + 1.

Given : $x(x + 1) = 20 \Rightarrow x^2 + x - 20 = 0$

$\Rightarrow (x + 5)(x - 4) = 0 \Rightarrow x = -5$, or $x = 4$

Since, x must be a natural number,

$\therefore x = 4$

And required numbers are x and x + 1 i.e., and 5.

Ex.26 The sum of the squares of two consecutive whole numbers is 61. Find the numbers.

Sol. Let the required consecutive whole numbers be x and x + 1.

$\therefore x^2 + (x + 1)^2 = 61$

$\Rightarrow x^2 + x^2 + 2x + 1 - 61 = 0 \Rightarrow 2x^2 + 2x - 60 = 0$

$\Rightarrow x^2 + x - 30 = 0$ [Dividing each term by 2]

$\Rightarrow (x + 6)(x - 5) = 0$ [On factorising]

$\Rightarrow x = -6$, or $x = 5$

$\because$ x is a whole number, $\therefore x = 5$

And, required numbers are x and x + 1 = 5 and 5 + 1 i.e., 5 and 6

Ex.27 The sum of two natural numbers is 8. If the sum of their reciprocals is $\frac{8}{15}$, find the two numbers.

Sol. Let the numbers be x and 8 – x.

$\therefore \frac{1}{x} + \frac{1}{8 - x} = \frac{8}{15} \Rightarrow \frac{8 - x + x}{x(8 - x)} = \frac{8}{15}$

$\Rightarrow \frac{8}{8x - x^2} = \frac{8}{15}$ i.e., $120 = 64x - 8x^2$

$\Rightarrow 8x^2 - 64x + 120 = 0$

$\Rightarrow x^2 - 8x + 15 = 0$ [Dividing by 8]

$\Rightarrow (x - 5)(x - 3) = 0$ [On factorizing]

$\Rightarrow x = 5$, or $x = 3$

When x = 5, the number are x and 8 – x = 5 and 3, and when x = 3, the numbers are x and 8 – x = 3 and 5.

$\therefore$ Required numbers are 5 and 3.

Ex.28 Divide 16 into two parts such that twice the square of the larger part exceeds the square of the smaller part by 164.

Sol. Let larger part be x, therefore the smaller part $= 16 - x$

Given : $2x^2 - (16 - x)^2 = 164$

$\Rightarrow\ 2x^2 - (256 + x^2 - 32x) - 164 = 0$

i.e., $2x^2 - 256 - x^2 + 32x - 164 = 0$

$\Rightarrow\ x^2 + 32x - 420 = 0$

On factorizing, it gives : $(x + 42)(x - 10) = 0$

i.e., $x = -42$ or $x = 10$

$\therefore\ x = 10$

Hence the larger part = 10 and the smaller part $= 16 - x = 16 - 10 = 6$

Ex.29 Two positive numbers are in the ratio 2 : 5. If difference between the squares of these numbers is 189 ; find the numbers.

Sol. Let numbers be 2x and 5x

$\therefore\ (5x)^2 - (2x)^2 = 189$

$\Rightarrow\ 25x^2 - 4x^2 = 189$ and $21x^2 = 189$

i.e., $x^2 = \dfrac{189}{21} = 9\ \Rightarrow\ x = \pm 3$

Since, the required numbers are positive,

$\therefore\ x = 3$

And, required numbers = 2x and $5x = 2 \times 3$ and $5 \times 3 = 6$ and 15

Ex.30 A two digit number is such that the product of the digits is 35. When 18 is added to this number the digits interchange their places. Determine the number.

Sol. Let ten's digit of the numbers = x and its unit digit = y.

$\therefore$ The two digit number is $10x + y$.

Given : $x \cdot y = 35$ and $10x + y + 18 = 10y + x$

$\Rightarrow\ y = \dfrac{35}{x}$ and $9x + 18 = 9y$

i.e., $x + 2 = y$

On substituting $y = \dfrac{35}{x}$ in $x + 2 = y$; we get:

$$x + 2 = \frac{35}{x}$$

$\Rightarrow x^2 + 2x = 35$

and $x^2 + 2x - 35 = 0$

On factorising, we get : $(x + 7)(x - 5) = 0$

i.e., $x = -7$ or $x = 5$

Since, x is digit, therefore $x = 5$ and

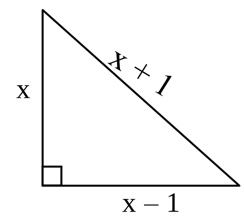
$$y = \frac{35}{x} = 7$$

$\therefore$ The required two digit number $= 10x + y$

$= 10 \times 5 + 7 = 57$

Ex.31 The sides (in cm) of a right triangle are $x - 1$, x and $x + 1$. Find the sides of triangle.

Sol. It is clear that the largest side $x + 1$ is hypotenuse of the right triangle.



According to Pythagoras Theorem, we have :

$x^2 + (x - 1)^2 = (x + 1)^2$

$\Rightarrow\ x^2 + x^2 - 2x + 1 = x^2 + 2x + 1$

This gives $x^2 - 4x = 0$

$\Rightarrow\ x(x - 4) = 0$ i.e., $x = 0$ or $x = 4$

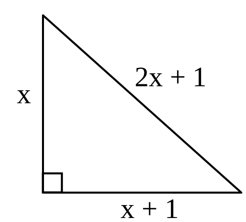
Since, with $x = 0$ the triangle is not possible; hence $x = 4$.

$\therefore$ Sides, are $x - 1$, x and $x + 1 = 4 - 1$

i.e., 3 cm, 4 cm and 5 cm

Ex.32 The hypotenuse of a right triangle is 1 m less than twice the shortest side. If the third side is 1 m more than the shortest side, find the sides of the triangle.

Sol. Let the shortest side be x m.



$\therefore$ Hypotensuse = $(2x - 1)$ m and the third side $= (x + 1)$ m

Applying Pythagoras theorem, we get ;

$$(2x - 1)^2 = x^2 + (x + 1)^2$$

$\Rightarrow$ $4x^2 - 4x + 1 = x^2 + x^2 + 2x + 1$

i.e., $2x^2 - 6x = 0$

$\Rightarrow$ $x^2 - 3x = 0$

i.e., $x(x - 3) = 0$ $\Rightarrow$ $x = 0$ or $x = 3$

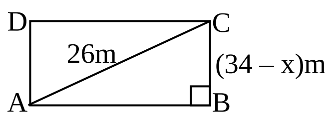
Since, x = 0 makes the triangle impossible. therefore, x = 3

And, sides of the triangle are = x, 2x – 1 and x + 1 = 3, 2 × 3 –1 and 3 + 1 = 3m, 5m

and 4m

Ex.33 If the perimeter of a rectangular plot is 68 m and its diagonal is 26 m. Find its area.

Sol. Let the length of plot = x m



$\because$ 2(length + breadth) = perimeter

$\Rightarrow$ 2(x + breadth) = 68

$\Rightarrow$ x + breadth $= \dfrac{68}{2}$ and breadth = (34 – x) m

Given its diagonal = 26 m and we know each angle of the rectangle = 90º.

$\therefore$ $x^2 + (34 - x)^2 = 26^2$

[Applying Pythagoras Theorem]

$\Rightarrow$ $x^2 + 1156 - 68x + x^2 - 676 = 0$

$\Rightarrow$ $2x^2 - 68x + 480 = 0$

$\Rightarrow$ $x^2 - 34x + 240 = 0$

i.e., $x^2 - 34x + 240 = 0$

On factorising, we get : $(x - 24)(x - 10) = 0$

i.e., x = 24 or x = 10

x = 24

$\Rightarrow$ length = 24 m and breadth

= (34 – 24) m = 10 m

and, x = 10

$\Rightarrow$ length = 10 m and breadth

= (34 – 10) m = 24 m

$\therefore$ Dimensions of the given rectangular plot are 24 m and 10 m.

Hence, its area = length × breadth

= 24 m × 10m = 240 m^2

Ex.34 A train travels a distance of 300 km at a uniform speed. If the speed of the train is increased by 5 km an hour, the journey would have taken two hours less. Find the original speed of the train.

Sol. Let the original speed of the train be x km/hr.

In 1st case, Distance = 300 km and speed = x km/hr.

$\Rightarrow$ Time taken $= \dfrac{\text{distance}}{\text{speed}} = \dfrac{300}{x}$ hrs.

In 2nd case, Distance = 300 km and

speed = (x +5) km/hr.

$\therefore$ Time taken $= \dfrac{\text{distance}}{\text{speed}} = \dfrac{300}{x+5}$ hrs.

Given : $\dfrac{300}{x} - \dfrac{300}{x+5} = 2$

$\Rightarrow$ $\dfrac{300(x+5) - 300x}{x(x+5)} = 2$

i.e., $\dfrac{300x + 1500 - 300x}{x^2 + 5x} = 2$

$\Rightarrow$ $2(x^2 + 5x) = 1500$

$\Rightarrow$ $x^2 + 5x - 750 = 0$

On factorising, we get : $(x + 30)(x - 25) = 0$

i.e., x = – 30 or x = 25

Neglecting x = – 30; we get x = 25

i.e., x = 25 km per hour

Ex.35 A motor boat, whose speed is 15 km/hr in still water, goes 30 km downstream and comes back in a total of 4 hours 30 minutes. Determine the speed of the stream.

Sol. Let the speed of the stream = x km/hr

$\Rightarrow$ The speed of the boat downstream = (15 + x) km/hr.

and, the speed of the boat upstream = (15 – x) km/hr

Now, time taken to go 30 km downstream

$= \dfrac{30}{15+x}$ hrs.

and, time take to come back 30 km upstream

$$= \frac{30}{15 - x} \text{ hrs.}$$

Given : the time taken for both the journeys

$$= 4 \text{ hours } 30 \text{ min.} = 4\frac{1}{2} \text{ hrs} = \frac{9}{2} \text{ hrs}$$

$$\therefore \quad \frac{30}{15 + x} + \frac{30}{15 - x} = \frac{9}{2}$$

$$\Rightarrow \quad \frac{30(15 - x) + 30(15 + x)}{(15 + x)(15 - x)} = \frac{9}{2}$$

$$\text{i.e.,} \quad \frac{450 - 30x + 450 + 30x}{225 - x^2} = \frac{9}{2}$$

$\Rightarrow$ $2 \times 900 = 9\ (225 - x^2)$

On dividing both the sides by 9, we get :

$2 \times 100 = 225 - x^2$

i.e., $x^2 = 225 - 200 \Rightarrow x^2 = 25$ and $x = \pm 5$

Rejecting the negative value of x,

we get : $x = 5$

i.e., the speed of the steam = 5 km/hr

Ex.38 Two years ago, a man's age was three times the square of his son's age. In three years time, his age will be four times his son's age. Find their present ages.

Sol. Let present age of son = x years

Two years ago : The age of son was $(x - 2)$ years and so the age of the man was $3(x - 2)^2$

$\therefore$ Man's present age = $3\ (x - 2)^2 + 2$

$= 3(x^2 - 4x + 4) + 2 = 3x^2 - 12x + 14$

In 3 years time : The age of son will be $(x + 3)$ years and the age of man will be

$(3x^2 - 12x + 14) + 3 = 3x^2 - 12x + 17$ years

Given : $3x^2 - 12x + 17 = 4\ (x + 3)$

$\Rightarrow 3x^2 - 12x + 17 = 4x + 12$ i.e., $3x^2 - 16x + 5 = 0$

On factorising, we get : $(x - 5)\ (3x - 1) = 0$

i.e., $x = 5$ or $x = \frac{1}{3}$

Since, $x = \frac{1}{3}$ is not possible ; $x = 5$

$\therefore$ The present age of man = $3x^2 - 12x + 14$

$= 3 \times 5^2 - 12 \times 5 + 14 = 29$ years.

And, the present age of son = x = 5 years

Ex.39 Find the roots of the equation $x^2 - 2x - 8 = 0$.

Sol. Quadratic Equation $x^2 - 2x - 8 = 0$

After factorization $(x - 4)\ (x + 2) = 0$

$\Rightarrow x = 4, -2$

Ex.40 Find the roots of the equation $x^2 - 4x + 1 = 0$.

Sol. Here a = 1, b = 4, c = 1

Using Hindu Method

$$x = \frac{4 \pm \sqrt{16 - 4}}{2} = 2 \pm \sqrt{3}$$

Ex.41 Find the nature of the roots of the quadratic equation $7x^2 - 9x + 2 = 0$.

Sol. $b^2 - 4ac = 81 - 56 = 25 > 0$ and a perfect square so roots are rational and different.

Ex.42 Find the nature of the roots of the quadratic equation $2x^2 - 7x + 4 = 0$.

Sol. $b^2 - 4ac = 49 - 32 = 17 > 0$ (not a perfect square) Its roots are irrational and different.

Ex.43 Find the nature of the roots of the quadratic equation $x^2 - 2\ (a + b)\ x + 2(a^2 + b^2) = 0$.

Sol. $A = 1, B = -2\ (a + b), C = 2\ (a^2 + b^2)$

$B^2 - 4AC = 1[2(a + b)]^2 - 4(1)\ (2a^2 + 2b^2)$

$= 4a^2 + 4b^2 + 8ab - 8a^2 - 8b^2$

$= -4a^2 - 4b^2 + 8\ ab$

$= -4(a-b)^2 < 0$

So roots are imaginary and different.

Ex.44 Find the nature of roots of the equation

$$x^2 - 2\sqrt{2}\ x + 1 = 0.$$

Sol. The discriminant of the equation

$(-2\sqrt{2})^2 - 4(1)\ (1) = 8 - 4 = 4 > 0$ and a perfect square so roots are real and different but we can't say that roots are rational because coefficients are not rational therefore.

$$\alpha, \beta = \frac{2\sqrt{2} - \sqrt{\left(2\sqrt{2}\right)^2 - 4}}{2}$$

$$= \frac{2\sqrt{2} \pm 2}{2} = \sqrt{2} \pm 1$$

this is irrational.

$\therefore$ the roots are real and different.

Ex.45 Find the nature of the roots of the equation

$(b + c) x^2 – (a + b + c) x + a = 0, (a,b,c \in Q)$?

Sol. The discriminant of the equation is

$(a +b + c)^2 – 4(b + c) (a)$

$= a^2+ b^2+ c^2 + 2ab + 2bc + 2ca – 4(b+c)a$

$= a^2 + b^2 + c^2 + 2ab + 2bc + 2ca – 4ab – 4 ac$

$= a^2 + b^2 + c^2 – 2 ab + 2 bc – 2 ca$

$(a – b – c)^2 > 0$

So roots are rational and different.

Ex.46 If the roots of the equation $x^2 + 2x +P = 0$ are real then find the value of P.

Sol. Here a = 1, b = 2, c = P

$\therefore$ discriminant = $(2)^2 – 4 (1) (P) 0$

(Since roots are real)

$\Rightarrow 4 – 4 P \geq 0 \Rightarrow 4 \geq 4P$

$\Rightarrow P \leq 1$

Ex.47 If the product of the roots of the quadratic equation $mx^2 – 2x + (2m–1) = 0$ is 3 then find the value of m is -

Sol. Product of the roots c/a = 3 = $\frac{2m-1}{m}$

$\therefore 3m – 2m = – 1 \Rightarrow m = – 1$

Ex.48 If α and β are roots of the equation

$x^2 – 5x + 6 = 0$ then find the value of $\alpha^3 + \beta^3$.

Sol. Here $\alpha + \beta = 5, \alpha\beta = 6$

Now $\alpha^3 + \beta^3 = (\alpha + \beta)^3 – 3\alpha\beta (\alpha + \beta)$

$= (5)^3 – 3.6 (5) = 125 – 90 = 35$

Ex.49 If the equation $(k – 2)x^2 – (k – 4) x – 2 = 0$ has difference of roots as 3 then find the value of k.

Sol. $(\alpha – \beta) = \sqrt{(\alpha+\beta)^2 - 4\alpha\beta}$

Now $\alpha + \beta = \frac{(k-4)}{(k-2)}, \alpha\beta = \frac{-2}{k-2}$

$\therefore (\alpha – \beta) = \sqrt{\left(\frac{k-4}{k-2}\right)^2 + \frac{8}{(k-2)}}$

$= \frac{\sqrt{k^2+16-8k+8(k-2)}}{(k-2)}$

$\Rightarrow 3 = \frac{\sqrt{k^2+16-8k+8k-16}}{(k-2)}$

$\Rightarrow 3k – 6 = \pm k$

$\therefore k = 3, 3/2$

Ex.51 Find the equation whose roots are 3 and 4.

Sol. The quadratic equation is given by

x^2 – (sum of the roots) x + (product of roots) = 0

$\therefore$ The required equation

$= x^2 – (3 + 4) x + 3.4 = 0$

$= x^2 – 7x + 12 = 0$

Ex.52 Find the quadratic equation with rational coefficients whose one root is $2 + \sqrt{3}$ -

Sol. The required equation is

$x^2 – \{(2+ \sqrt{3}) + (2– \sqrt{3}) \} x + (2 + \sqrt{3}) (2– \sqrt{3}) = 0$

or $x^2 – 4x + 1 = 0$

Ex.53 If α, β are the root of a quadratic equation $x^2 – 3x + 5 = 0$ then find the equation whose roots are $(\alpha^2 – 3\alpha + 7)$ and $(\beta^2 – 3\beta + 7)$.

Sol. Since α, β are the roots of equation

$x^2 – 3x + 5 = 0$

So $\alpha^2 – 3\alpha + 5 = 0$ & $\beta^2 –3\beta + 5 = 0$

$\therefore \alpha^2 – 3\alpha = – 5$ & $\beta^2 – 3\beta = – 5$

putting in $(\alpha^2 – 3\alpha + 7)$ & $(\beta^2 – 3\beta + 7)$ ...(1)

$–5 + 7 , – 5 + 7$

2 and 2 are the roots

the required equation is

$x^2 – 4x + 4 = 0.$

Ex.54 If α,β are roots of the equation $x^2 – 5x + 6 = 0$ then find the equation whose roots are $\alpha + 3$ and $\beta + 3$ is -

Sol. Let $\alpha + 3 = x$

$\therefore \alpha = x – 3$ (Replace x by x – 3)

So the required equation is

$(x-3)^2 - 5(x-3) + 6 = 0$ ...(1)

$\Rightarrow$ $x^2 - 6x + 9 - 5x + 15 + 6 = 0$

$\Rightarrow$ $x^2 - 11x + 30 = 0$ ...(2)

Ex.55 If α, β are roots of the equation $2x^2 + x - 1 = 0$ then find the equation whose roots are $1/\alpha$, $1/\beta$ will be -

Sol. From the given equation

$\alpha + \beta = -1/2$, $\alpha\beta = -1/2$

The required equation is -

$$x^2 - \left(\frac{1}{\alpha} + \frac{1}{\beta}\right) x + \frac{1}{\alpha\beta} = 0$$

$$\Rightarrow x^2 - \left(\frac{\alpha+\beta}{\alpha\beta}\right) x + \frac{1}{\alpha\beta} = 0$$

$$\Rightarrow x^2 - \left(\frac{-\frac{1}{2}}{-\frac{1}{2}}\right) x + \frac{1}{-\frac{1}{2}} = 0 \Rightarrow x^2 - x - 2 = 0$$

Short cut : Replace x by 1/x

$\Rightarrow$ $2(1/x)^2 + 1/x - 1 = 0$ $\Rightarrow$ $x^2 - x - 2 = 0$

Ex.56 If r and s are positive, then find the nature of roots of the equation $x^2 - rx - s = 0$

Sol. Here Discriminant

$= r^2 + 4s > 0$ $(\because r, s > 0)$

$\Rightarrow$ roots are real

Again $a = 1 > 0$ and $c = -s < 0$

$\Rightarrow$ roots are of opposite signs.

Ex.58 If one root of the equation $4x^2 + 2x - 1 = 0$ is α, then find the other root.

Sol. Let α and β are roots of the given equation, then $\alpha + \beta = -\frac{1}{2}$ $\Rightarrow \beta = -\frac{1}{2} - \alpha$

Now $4\alpha^2 + 2\alpha - 1 = 0$

$4\alpha^2 = 1 - 2\alpha$ ...(1)

Now $4\alpha^3 = \alpha - 2\alpha^2$

$= \alpha - \frac{1}{2}(1 - 2\alpha)$ [from (1)]

$\therefore$ $4\alpha^3 - 3\alpha = -2\alpha - \frac{1}{2}(1 - 2\alpha)$

$= -\frac{1}{2} - \alpha = \beta$

Ex.59 Find the quadratic equation, whose one root is $\frac{1}{2+\sqrt{5}}$.

Sol. Given root $= \frac{1}{2+\sqrt{5}} = \sqrt{5} - 2$

So the other root $= -\sqrt{5} - 2$. Then sum of the roots $= -4$, product of the roots $= -1$

Hence the equation is $x^2 + 4x - 1 = 0$